

Contents

1	Introduction	1
1.1	Intended Readership	8
1.2	Notation and Terminology	9
2	Networks: Introductory Material	17
2.1	Basic Network Definitions	17
2.2	Small-World Networks	25
2.3	Random Networks	32
2.4	Scale-Free Networks	33
2.5	Skeleton of a Network	40
3	Fractals: Introductory Material	43
3.1	Introduction to Fractals	43
3.2	Fractals and Power Laws	49
3.3	Iterated Function Systems	51
3.4	Convergence of an IFS	54
4	Topological and Box Counting Dimensions	61
4.1	Topological Dimension	61
4.2	Box Counting Dimension	67
4.3	A Deeper Dive into Box Counting	73
4.4	Fuzzy Fractal Dimension	77
4.5	Minkowski Dimension	78
5	Hausdorff, Similarity, and Packing Dimensions	83
5.1	Hausdorff Dimension	84
5.2	Similarity Dimension	89
5.3	Fractal Trees and Tilings from an IFS	96
5.4	Packing Dimension	99
5.5	When Is an Object Fractal?	103

6	Computing the Box Counting Dimension	107
6.1	Lower and Upper Box Sizes	107
6.2	Computing the Minimal Box Size	111
6.3	Coarse and Fine Grids	115
6.4	Box Counting Dimension of a Spiral	119
6.5	Some Applications of the Box Counting Dimension	122
7	Network Box Counting Dimension	131
7.1	Node Coverings and Arc Coverings	132
7.2	Diameter-Based and Radius-Based Boxes	135
7.3	A Closer Look at Network Coverings	140
7.4	The Origin of Fractal Networks	143
8	Network Box Counting Heuristics	145
8.1	Node Coloring Formulation	145
8.2	Node Coloring for Weighted Networks	148
8.3	Random Sequential Node Burning	150
8.4	Set Covering Formulation and a Greedy Method	154
8.5	Box Burning	157
8.6	Box Counting for Scale-Free Networks	161
8.7	Other Box Counting Methods	163
8.8	Lower Bounds on Box Counting	165
9	Correlation Dimension	177
9.1	Dynamical Systems	177
9.2	Pointwise Mass Dimension	182
9.3	The Correlation Sum	183
9.4	The Correlation Dimension	186
9.5	Comparison with the Box Counting Dimension	191
10	Computing the Correlation Dimension	195
10.1	How Many Points are Needed?	195
10.2	Analysis of Time Series	201
10.3	Constant Memory Implementation	204
10.4	Applications of the Correlation Dimension	205
10.5	The Sandbox Dimension	217
11	Network Correlation Dimension	221
11.1	Basic Definitions	222
11.2	Variants of the Correlation Dimension	226
11.3	Cluster Growing and Random Walks	230
11.4	Overall Slope Correlation Dimension	234
11.5	Comparing the Box Counting and Correlation Dimensions	243
11.6	Generalized Correlation Dimension	245

12	Dimensions of Infinite Networks	247
12.1	Mass Dimension	249
12.2	Transfractal Networks	253
12.3	Volume and Surface Dimensions	256
12.4	Conical Graphs with Prescribed Dimension	262
12.5	Dimension of a Percolating Network	265
13	Similarity Dimension of Infinite Networks	267
13.1	Generating Networks from an IFS	270
13.2	Analysis of the Infinite Sequence	272
14	Information Dimension	279
14.1	Shannon Entropy and Hartley Entropy	280
14.2	Dimension of a Probability Distribution	286
14.3	Information Dimension Bounds the Correlation Dimension	290
14.4	Fixed Mass Approach	295
14.5	Rényi and Tsallis Entropy	297
14.6	Applications of the Information Dimension	300
15	Network Information Dimension	305
15.1	Entropy of a Node and a Network	305
15.2	Defining the Network Information Dimension	311
15.3	Comparison with the Box Counting Dimension	317
15.4	Maximal Entropy Minimal Coverings	318
15.5	Binning	324
16	Generalized Dimensions and Multifractals	325
16.1	Multifractal Cantor Set	327
16.2	Generalized Dimensions	329
16.3	Spectrum of the Multifractal Cantor Set	333
16.4	Spectrum of a Multifractal	338
16.5	Computing the Multifractal Spectrum and Generalized Dimensions	346
16.6	The Sandbox Method for Geometric Multifractals	353
16.7	Particle Counting Dimension	359
16.8	Applications of Multifractals	361
16.9	Generalized Hausdorff Dimensions	363
17	Multifractal Networks	365
17.1	Lexico Minimal Coverings	365
17.2	Non-monotonicity of the Generalized Dimensions	373
17.3	Multifractal Network Generator	383
17.4	Computational Results for Multifractal Networks	386
17.5	Sandbox Method for Multifractal Networks	388

18	Generalized Hausdorff Dimensions of Networks	391
18.1	Defining the Hausdorff Dimension of a Network	391
18.2	Computing the Hausdorff Dimension of a Network	396
18.3	Generalized Hausdorff Dimensions of a Network	407
19	Lacunarity	413
19.1	Gliding Box Method	415
19.2	Lacunarity Based on an Optimal Covering	420
19.3	Applications of Lacunarity	421
20	Other Dimensions	425
20.1	Spectral Dimension	425
20.2	Lyapunov Dimension	427
20.3	Three More Dimensions	429
20.4	Uses and Abuses of Fractal Dimensions	431
21	Coarse Graining and Renormalization	437
21.1	Renormalization Basics	438
21.2	Renormalizing a Small-World Network	442
21.3	Renormalizing Unweighted, Undirected Networks	445
21.4	Renormalization of Scale-Free Networks	448
21.5	Universality Classes for Networks	452
22	Other Network Dimensions	455
22.1	Dimension of a Tree	455
22.2	Area-Length Dimension	457
22.3	Metric Dimension	458
22.4	Embedding Dimension	459
22.5	Zeta Dimension	461
22.6	Dimensions Relating to Graph Curvature	465
22.7	Spectral Dimension of Networks	467
22.8	Other Avenues	468
23	Supplemental Material	471
23.1	Rectifiable Arcs and Arc Length	471
23.2	The Gamma Function and Hypersphere Volumes	473
23.3	Laplace's Method of Steepest Descent	474
23.4	Stirling's Approximation	475
23.5	Convex Functions and the Legendre Transform	476
23.6	Graph Spectra	480
	Bibliography	485
	Index	517